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THE PROBLEM OF THE EIGENVALUES AND MODES OF ROTATING DEFORMABLE STRUCTURES†

I. YE. TROYANOVSKII, I. N. SHARDAKOV and N. A. SHEVELEV

Moscow, Perm

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A formulation analogous to that discussed earlier in [1] is used as the framework for constructing an algorithm, based on the finite-element method, for the numerical solution of the problem of determining the eigenvalues and mode of rotating, dynamically symmetrical bodies. It is proposed to seek the eigen modes of the nonconservative problem in the form of an expansion in eigen modes of the conservative problem, which reduces the dimensions of the matrices and enables the complex problem of the eigenvalues to be solved with the help of well-known and tested methods.

THE CHARACTERISTIC features of the rigidity of rotating structures can change under the action of centrifugal force. If the terminology adopted in [1] is used, then, when the angular velocity is constant, one is dealing with a system in which the number of rotations can be controlled and this represents a typical nonconservative system. This means that in order to obtain complete information about the "dynamic record of the system" one must consider, in addition to the problem of the eigen frequencies and modes of the oscillations of elastic rotating bodies, also the problem of nonconservative elastic stability.

In many cases, and especially when solving examples, one inspects the stability using the first-approximation equations, and the effect of the initial stress state on the eigen modes and frequencies of the oscillations is found from the solution of the linearized equations of motion [2]. The determination of eigenvalues is the common feature of such approaches. The final conclusions concerning the stability can be based on the form of the eigenvalues obtained, for example, using the framework of the Lyapunov theorems on stability in the first approximation, or by showing that these eigenvalues correspond to the oscillation eigenfrequencies.

This paper deals with the problem of the eigenvalues and eigen modes of the motion of a nonconservative system. Depending on the geometry of the structure, material properties, boundary conditions and other parameters defining the system, the eigenvalues will be imaginary, real, or complex.

The usual methods of computing the eigenfrequencies and modes of the oscillations applicable to conservative systems, for example [3], cannot be used for the systems discussed here [4]. It is therefore necessary to stipulate the possible changes in existing approaches and alorithms which have shown themselves to be successful in solving the conservative problems, and supplement them by a number of new elements. For example, the procedure of the reverse iteration method [2] and the method of parabolas in complex form [5] will after combining them with the semi-analytic method of finite elements [6], satisfy the needs of the problems considered.

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1. FORMULATION OF THE PROBLEM

An elastic body occupies, in three-dimensional Euclidean space, a volume V bounded by the surface Σ. Boundary conditions are specified on part of the surface Σ_u in displacements, and on the remaining part of the surface Σ_σ in the stresses. The body rotates with angular velocity ω of constant magnitude, about an axis which coincides with the axis of symmetry of the body. The relative Coriolis and centripetal accelerations related to elastic deformations of the body are taken into account. The influence of stationary internal stresses due to the action of centripetal forces are not discussed. The dynamic characteristics of the system, the eigenvalues and eigen modes of the motion appearing near the stationary position are all to be determined. In other words, it is necessary to determine the functional dependence of the perturbations on time.

The mathematical formulation of the problem includes the equations of motion [7, 8]

$$(\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} + \mu\Delta\mathbf{u} = \rho\,[\mathbf{u}^{\cdot\cdot} + 2\omega \times \mathbf{u}^{\cdot} + \omega \times (\omega \times \mathbf{u})], \quad r, z, \varphi \in V \tag{1.1}$$

where $\mathbf{u}$ is the displacement vector in a rotating cylindrical system of coordinates r, z, φ whose z axis is directed along the vector ω, σ is the stress tensor in the cylindrical system of coordinates, ρ is the density of the material of the body, the boundary conditions in displacements on the part of the surface Σ_u and in stresses on the part of the surface $\Sigma_\sigma = \Sigma_u + \Sigma_\sigma$, $\boldsymbol{\nu}$ is the vector of outward normal to the surface Σ)

$$\mathbf{u}\,(r, z, \varphi, t) = 0, \quad r, z, \varphi \in \Sigma_u \qquad \nu \cdot \sigma\,(\mathbf{u}) = 0, \quad r, z, \varphi \in \Sigma_\sigma \tag{1.2}$$

In the case of an isotropic linear material the stress and strain tensor components are connected by the following physical relations:

$$\sigma = \lambda\Theta E + 2\mu\varepsilon \tag{1.3}$$

Here λ, μ are the Lamé constants, Θ is the first invariant of the strain tensor and E is the unit tensor of second rank. The following Cauchy relations hold for the strain tensor and displacement vector components:

$$\varepsilon = {}^1/_2\,[(\nabla\mathbf{u})^T + \nabla\mathbf{u}] \tag{1.4}$$

The unknown displacement vector can be represented, in the case in question, in the form

$$\mathbf{u}\,(r, z, \varphi, t) = U\,(r, z, \varphi)\, e^{pt} \tag{1.5}$$

where p is the complex eigenvalue, to be regarded as eigenfrequency of oscillations or the stability parameter.

After substituting expression (1.5) into Eqs (1.1) and its boundary conditions (1.2), we obtain the following boundary value problem for determining the vector $\mathbf{U}$ which represents a mode of the oscillations if p is an imaginary number, or a mode of loss of stability if the real part of the complex number p or the real p is greater than zero:

$$\begin{gathered}(\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{U} + \mu\Delta\mathbf{U} = \rho\,[p^2\mathbf{U} + 2p\boldsymbol{\omega} \times \mathbf{U} + \boldsymbol{\omega} \times (\omega \times \mathbf{U})],\\ r, z, \varphi \in V; \quad \mathbf{U}\,(r, z, \varphi) = 0, \; r, z, \varphi \in \Sigma_u \\ \boldsymbol{\nu} \cdot \{\lambda\Theta E + \mu\,[(\nabla\mathbf{U})^T + (\nabla\mathbf{U})]\} = 0, \quad r, z, \varphi \in \Sigma_\sigma\end{gathered} \tag{1.6}$$

When using numerical methods to determine the displacement vector, it is more convenient to use the variational formulation

$$\delta A_\sigma + \delta A_i + \delta A_c + \delta A_k = 0 \tag{1.7}$$
$$\delta A_\sigma = -\int \sigma \cdot \delta\varepsilon \, dv, \quad \delta A_i = -\rho p^2 \int \mathbf{U} \cdot \delta\mathbf{U} \, dv$$
$$\delta A_c = -\rho \int [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{U})] \cdot \delta\mathbf{U} \, dv$$
$$\delta A_k = -2\rho p \int (\boldsymbol{\omega} \times \mathbf{U}) \cdot \delta\mathbf{U} dv$$

Here δA_σ is the work done by internal stresses determined by the possible displacements, δA_i is the work done by inertial forces, δA_c is the work done by centripetal forces connected with the displacement field $\mathbf{U}$, δA_k is a term determined by Coriolis forces and integration is carried out over the volume V.

The unknown displacement vector $\mathbf{U}$ and its variation $\delta\mathbf{U}$ must satisfy the boundary conditions in the displacements, while the conditions in the stresses are satisfied automatically.

2. NUMERICAL SOLUTION

The geometrical symmetry of the bodies under discussion about the axis of rotation, enables us to write the unknown vector $\mathbf{U}(r, z, \varphi)$ in the form of an expansion in the circumferential coordinate φ

$$\mathbf{U} = \begin{Vmatrix} u \\ w \\ v \end{Vmatrix} = \begin{Vmatrix} \sum u_n{}^s(r, z) \cos n\varphi + u_n{}^a(r, z) \sin n\varphi \\ \sum w_n{}^s(r, z) \cos n\varphi + w_n{}^a(r, z) \sin n\varphi \\ \sum v_n{}^s(r, z) \sin n\varphi + v_n{}^a(r, z) \cos n\varphi \end{Vmatrix} \tag{2.1}$$

Here u, w, v are the components of the vector $\mathbf{U}$ in coordinates r, z, φ, respectively, the superscript s corresponds to the symmetric component and a to the antisymmetric component of the vector relative to the radius $\varphi = 0$, and summation is carried out from $n = 1$ to $n = N$.

We can write the displacement vector in symbolic form as follows:

$$\mathbf{U} = \mathbf{U}_s + \mathbf{U}_a \tag{2.2}$$

If we disregard the terms δA_c and δA_k in the variational equation, the terms related to the ($\varphi = 0$), the problem of determining $\mathbf{U}$ will separate into two independent problems for $\mathbf{U}_s$ and $\mathbf{U}_a$, and the eigenvalues p_s and p_a corresponding to the eigenvectors, will be identical. In the case of a nonconservative system the forms $\mathbf{U}_s$ and $\mathbf{U}_a$ are interrelated and can be obtained from the general equation (1.7), and the vector $\mathbf{U}$ will represent their superposition. We will adopt, for the variation $\delta\mathbf{U}$, an expansion analogous to (2.1). Evaluating the integrals in (1.7), taking the representation $\mathbf{U}$ and $\delta\mathbf{U}$ into account in the form (2.1) as well as the orthogonality properties of trigonometric functions, we obtain a variational problem for the separate harmonics n in expansion (2.1).

We construct the vectors $\mathbf{U}_s$ and $\mathbf{U}_a$ numerically using the finite-element method. We will give the basic relations of the semi-analytic finite element method retaining the notation adopted in [6].

We divide the cross-section of the solid of revolution into triangular elements, containing at each node six displacement components with a linear approximation of the latter within the element, three displacements corresponding to the symmetric displacement vector, and three corresponding to the antisymmetric vector. Thus every element will contain 18 nodal displacements. In matrix form we have

$$\mathbf{u}^e = \mathbf{N}^e \delta^e \tag{2.3}$$

where $\mathbf{U}^e$ is the displacement vector in the element, $\mathbf{N}^e$ is the matrix of the modal functions of dimension 3×18 and $\boldsymbol{\delta}^e$ is the column vector of nodal displacements. We can write analogous matrix relations for the physical equations and Cauchy relations:

$$\boldsymbol{\sigma}^e = \mathbf{D}\boldsymbol{\varepsilon}^e, \ \boldsymbol{\varepsilon}^e = \mathbf{B}\boldsymbol{\delta}^e \tag{2.4}$$

where the matrix $\mathbf{B}^e$ is determined taking (2.3) into account and $\mathbf{D}$ is the matrix of elastic constants.

Taking into account the notation given above, we shall write the finite-element analogue of the variational equation (1.7) in the form

$$\begin{gathered}(\mathbf{K} + \rho p^2 M + \rho\omega^2 \mathbf{F}_c + 2\omega\rho p \mathbf{F}_k)\,\boldsymbol{\delta} = 0 \\ \mathbf{K} = \int \mathbf{B}^T \mathbf{D}\mathbf{B}\,dv, \quad \mathbf{M} = \int \mathbf{N}^T \mathbf{N}\,dv \\ \mathbf{F}_c = \int \mathbf{N}^T \mathbf{f}_c\,dv, \quad \mathbf{F}_k = \int \mathbf{N}^T \mathbf{f}_k\,dv\end{gathered} \tag{2.5}$$

where $\mathbf{K}$, $\mathbf{M}$, $\mathbf{F}_c$, $\mathbf{F}_k$ are the matrices of rigidity, mass, the centripetal and Coriolis forces, respectively. The matrix $\mathbf{f}_c$ is given in terms of the modal functions of the element and is symmetric, while the matric $\mathbf{f}_k$ is antisymmetric. Making the change of variables

$$t_0 = t\Omega^{-1}, \quad \partial^2/\partial t_0^2 = \Omega\partial^2/\partial t^2 \tag{2.6}$$

and putting $\Omega = \omega$, we can rewrite the Eq. (2.5) in the form

$$(\mathbf{K}^0 + p_0^2 \mathbf{M}^0 + p_0 \mathbf{F}_k^0)\,\boldsymbol{\delta} = 0 \tag{2.7}$$

where the matrices with zero superscript differ from the corresponding matrices from (2.5) in the multipliers and $p = p_0\Omega$. Such a change of the variable t and choice of Ω leads to normalization of the numerical value of the elements of the matrices $\mathbf{K}$, $\mathbf{M}$, $\mathbf{F}_c$, $\mathbf{F}_k$ and to their equivalent contribution towards the resulting coefficients of system (2.7).

The complex eigenvalues p_0 appearing in the system of linear algebraic equations as the unknown parameters, are determined using the method of parabolas [5] in complex form. We shall call such a scheme for solving the problem in question the direct method of determining eigenvalues. Its drawbacks include the high order of the system of equations (2.7) and the low efficiency of the method of parabolas in complex form for matrices of large dimensions.

In order to reduce the dimensions of the system, we shall use the method of expansion in eigen modes of the corresponding problem for a conservative system. The latter problem is real and therefore much simpler than the complex problem (2.7).

The expansion contains linear combinations of a finite number of first eigen modes of the oscillations of the corresponding elastic problem without rotation [9]

$$\boldsymbol{\delta} = \sum_{k=1}^{m} q_k \chi_k, \quad q_k = \text{const} \tag{2.8}$$

where λ_k is the eigenfrequency of the oscillations of the conservative system, and χ_k is the eigen mode of the oscillations corresponding to λ_k. When (2.8) is taken into account, system (2.7) will become

$$(\mathbf{K} - \rho\lambda_k^2 \mathbf{M})\,\chi_k = 0 \tag{2.9}$$

where λ_k is the eigenfrequency of the oscillations of the conservative system, and χ_k is the eigen mode of the oscillations corresponding to λ_k. When (2.8) is taken into account, system (2.7) will become

$$(\mathbf{K}' + p_0^2 \mathbf{M}' + p_0 \mathbf{F}_k')\,\mathbf{q} = 0 \tag{2.10}$$

The matrices in (2.10) have dimensions $2m \times 2m$, and their elements are given by the relations

$$K_{ij}' = \chi_i^T \mathbf{K}^0 \chi_j, \quad M_{ij}' = \chi_i^T \mathbf{M}^0 \chi_j \tag{2.11}$$
$$F'_{(k)ij} = \chi_i^T \mathbf{F}_k^0 \chi_j, \quad i, j = 1, 2, \ldots, m$$

The dimensions of the matrices are twice the number of terms retained in the expansion, due to the need to take into account the symmetric and antisymmetric components in the vectors χ_k.

The method of expansion in terms of the eigen modes is convenient in the case when the properties of the structural material and the angular velocity parameter vary, in which case the modes of the oscillations determined earlier can be used time and again to investigate the stability. It is however difficult to obtain in advance the necessary number of eigen modes in the expansion (2.8). Computations carried out show that this drawback can be compensated by a substantial increase in the efficiency of the method of determining the eigenvalues, and by carrying out the numerical experiment with a different number of modes retained in the expansion one can guarantee the correctness of the results obtained.

3. EXAMPLES OF COMPUTATIONS

Numerical computations were carried out for an elastic body in the form of a disc of following geometrical dimensions: inner radius 0.02 m, outer radius 0.125 m, thickness 0.01 m, rigidly clamped along the inner contour, with a Poisson's ratio of 0.26. The outer contour of the disc is load free. In order to establish the usefulness of the proposed algorithms, the problem was solved using the direct method and method of expansion in eigen modes of the conservative problem. Moreover, the results obtained constructing the eigen modes of the oscillations of the conservative problem make it possible to determine the necessary number of terms in expansion (2.8).

We shall illustrate this using a specific example. The values of the first seven eigenfrequencies of the oscillation of the disc (in Hertz) without rotation for $E = 2 \times 10^{11}$ N/m^2 and $\rho = 7830$ kg/m^3 were as follows: 74, 564, 1614, 3173, 5245, 5944, 7824. All results quoted in the paper refer to the first harmonic in the expansion in the angular coordinate. Flexural modes of oscillations correspond to the first five eigenfrequencies, the sixth frequency has the mode of oscillations in the plane, the seventh is again flexural, etc. (alternation of the flexural and plane forms of the oscillations). Their sequential order may change for different geometrical dimensions and boundary conditions.

The additional forces of inertia caused by rotation have a different effect on the change in the eigenvalues quantitatively, as well as qualitatively. The eigenvalues to which the flexural modes of the oscillations correspond may decrease as the angular velocity increases and remain purely imaginary (Fig. 1, p_1) until one of them vanishes. When ω is increased further the quantity p_1, for example, will become real and positive, and such eigenvalues will vary quite slowly. According to investigations [1, 10] such an instability is called a static instability. In order to show how the eigenvalues of the corresponding conservative problems change when rotation is taken into account, we will give the values of the eigenfrequencies at $E = 2 \times 10^{11}$ N/m^2 and $\rho = 7830$ kg/m^3 for an angular velocity of $\omega = 5000$ rpm: 56.4, 56.9, 547.2, 547.6, 1601.2, 1600.8, 3160.9, 3161.4, 5232.9, 5233.4, $(1274.0 + 3547.8i)$, $(1274.0 - 3547i)$, 7813.0. The sixth eigenvalue corresponding to a plane mode of the oscillations corresponds to the change in the angular velocity in a completely different manner than the flexural frequencies mentioned above.

We note that in the case when $\omega = 0$ we have multiple eigenfrequencies which correspond to the symmetric and antisymmetric modes of motions. When an angular velocity appears, they begin to change quantitatively according to a very complicated rule, but remaining purely imaginary (Fig. 1). Beginning from some value of the parameter ω, they converge and form a pair of complex eigenvalues with a positive real part. This effect was described in [1] as an oscillatory instability.

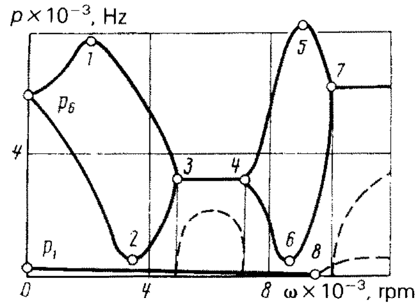

FIG. 1.

The results were obtained using the direct method and the method of expansions in eigen modes. Comparisons have shown that the eigenvalues are practically identical in both cases when the expansion contains 16 eigen modes of the oscillations.

Table 1 gives normalized coefficients q_k in expansion (2.8), the superscripts s and a are as before, and the numbers 1, 2, . . . , 16 correspond to the order of the frequency sequence in the spectrum. The flexural modes have the numbers 1, 2, 3, 4, 5, 7, 8, 10, 11, and the plane modes have the numbers 6, 9, 13, 16. Analysis of the table enables us to form the sequence of the functions χ_k and enables us to use them further in the expansion.

TABLE 1.

Mode	Frequency					
	$p_1 = 56.4$	$p_2 = 56.9$	$p_3 = 547.2$	$p_4 = 547.6$	$p_{10} = 5233.4$	$p_{11,12} = 1274 \pm 3547i$
q_k^s	$1.0+0i$ $0.001+0i$ $-0.0008+0i$ 0	0	$-0.01+0i$ $1.0+0i$ $-0.005+0i$ $0.0008+0i$ 0	0	0	$0, k = 1, 2, \ldots, 5$ $1.0+0i, k = 6$ $0, k = 7, 8$ $0.294-0.226i, k = 9$ $0, k = 10, 11, 12$ $0.027-0.021i, k = 13$ $0, k = 14, 15$ $-0.002+0.003i, k = 6$
q_k^a	0	$1.0+0i$ $0.01+0i$ $-0.008+0i$ 0	0	$-0.01+0i$ $1.0+0i$ $-0.005+0i$ $0.008+0i$ 0	0 $0.0004+0i$ $-0.0004+0i$ $0.001+0i$ $1.0+0i$ $0.001+0i$ $0.002+0i$ 0	$0, k = 1, 2, \ldots, 5$ $0+1.0i, k = 6$ $0, k = 7, 8$ $-0.226+0.294i, k = 9$ $0, k = 10, 11, 12$ $-0.021+0.027i, k = 13$ $0, k = 14, 15$ $0.003-0.002i, k = 16$

Figure 1 ($E = 2\times10^{11}$ N/m^2, $\rho = 7830$ kg/m^3) and similar to it Fig. 2 ($E = 10^{11}$ N/m^2, $\rho = 7830$ kg/m^3) and Fig. 3 ($E = 2\times10^{11}$ N/m^2, $\rho = 4540$ kg/m^3) show the graphs of the change of the first and sixth eigenvalue as a function of the angular velocity parameter. The thick lines depict the imaginary part of the complex eigenvalues, and the dashed lines correspond to the real part.

When formulating the problem, we remarked that the eigen oscillation frequencies are affected by the initial state of stress. In the case of a conservative problem this effect was studied in [2]. Since in the present paper the analysis is carried out without taking this fact into account, we constructed, as a tentative estimate, graphs of the changes in the characteristics of the eigenvalues (Figs 2 and 3) for discs with different material

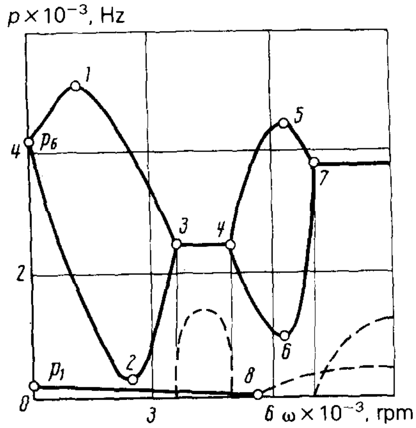

FIG. 2.

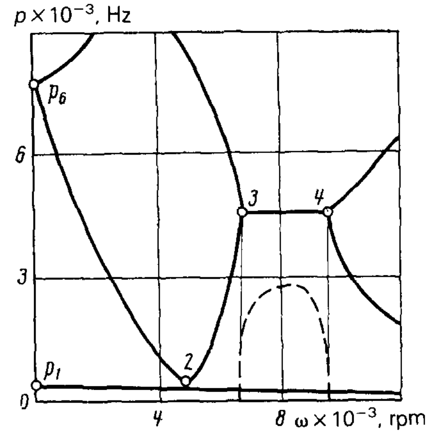

FIG. 3.

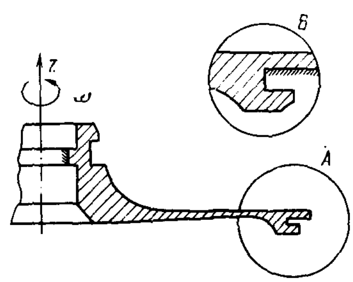

FIG. 4.

characteristics, which models to some degree the change in rigidity caused by the initial state of stress. Comparing Figs 1–3 we can conclude that inclusion of the initial state of stress in our discussion does not change the qualitative pattern of the appearance of instability; it merely shifts the domain of instability towards an increase or decrease in the critical angular velocity.

The proposed approach was used to study the stability of an airplane deflector (Fig. 4). The quantitative results are identical with those given in Figs 1–3. The first zones of instability for a structure with type A boundary conditions were established within the range 500 rpm$<\omega^*<$800 rpm, and for type B conditions within the range 4200 rpm$<\omega^*<$5500 rpm.

A complete study of the dynamic behaviour of structures requires that the above analysis be carried out for several harmonics in the expansion in the angular coordinate.

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STABILIZATION OF HOLONOMIC CONTROLLED SYSTEMS NEAR A POSITION OF EQUILIBRIUM†

B. N. SOKOLOV

Moscow

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THIS PAPER is a continuation of the study of various laws of positional control of large dynamic systems [1–3], and deals with a holonomic controlled system near its position of equilibrium. The necessary and sufficient conditions for the existence of a control ensuring asymptotic stability of the system as a whole are obtained. A structure of the control, which is the simplest in a certain sense, which solves the problem in question, is given.

Let M, C, P, G be the matrices of mass, dissipative forces, potential energy and control, respectively, q the vector of generalized coordinates, and u the control. C and P are non-negative definite matrices, and M is a positive definite matrix. The motion of a holonomic system near the position of equilibrium is described by the equations [4]

$$Mq^{\cdot\cdot} + Cq^{\cdot} + Pq = Gu, \quad q \in R^n, \quad u \in R^m \tag{1}$$

Linear controlled systems of general type were studied in sufficient detail in [4, 5], and corresponding methods for obtaining a control for solving two-point boundary value problem were developed. If the system is of large dimensions the construction of positional control taking the system to a prescribed position is difficult. Therefore, regulators are often used which ensure the asymptotic stability of the dynamic system in the required position [5]. Suppose the system in question is of large dimensions, and it is required to construct a regulator which depends on the minimum number of generalized coordinates. Below we obtain the necessary and sufficient conditions determining the control matrix of such a regulator and the corresponding control is given.

We shall call the subspace L on which the non-negative definite form vanishes, the null subspace. We will denote by L_1 and L_2 the null subspaces of quadratic forms q^TCq and q^TPq (1) respectively.

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